Strongly Correlated Phases in Rapidly Rotating Bose Gases

Mathieu Lewin · Robert Seiringer

Abstract We consider a system of trapped spinless bosons interacting with a repulsive potential and subject to rotation. In the limit of rapid rotation and small scattering length, we rigorously show that the ground state energy converges to that of a simplified model Hamiltonian with contact interaction projected onto the Lowest Landau Level. This effective Hamiltonian models the bosonic analogue of the Fractional Quantum Hall Effect (FQHE). For a fixed number of particles, we also prove convergence of states; in particular, in a certain regime we show convergence towards the bosonic Laughlin wavefunction. This is the first rigorous justification of the effective FQHE Hamiltonian for rapidly rotating Bose gases. We review previous results on this effective Hamiltonian and outline open problems.

1 Introduction

A fundamental characteristic of trapped Bose gases is their response to rotation [11, 15]. When the angular velocity Ω becomes large, a transition from a condensed regime to a highly correlated, uncondensed, phase is expected. The behavior of the system then has certain features similar to the Fractional Quantum Hall Effect that is observed in superconductors submitted to a magnetic field. For rotating Bose gases, this regime has not yet been observed experimentally, the corresponding value of Ω being unattainable at present. Nevertheless, there has been a lot of interest in the theoretical understanding of this phenomenon in the literature (see, e.g., [11] for a recent review).

M. Lewin
CNRS & Laboratoire de Mathématiques UMR 8088, Université de Cergy-Pontoise, 2 Avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France
e-mail: Mathieu.Lewin@math.cnrs.fr

R. Seiringer (✉)
Department of Physics, Princeton University, Princeton, NJ 08544, USA
e-mail: rseiring@princeton.edu

In this paper we present a rigorous study of such a Bose system with a generic repulsive interaction potential. We show that in a certain limit, the Hamiltonian of the system can be replaced by a simplified effective Hamiltonian with a contact interaction, projected onto the Lowest Landau Level (LLL). This simplified model has been extensively used in the physics literature [13–15, 29, 30, 32, 36]. We also prove that in the same limit, the true ground state of the system converges to the ground state of the model Hamiltonian. In a certain parameter regime, our analysis provides a rigorous derivation of the bosonic equivalent of the well-known Laughlin state [20].

Let us consider N interacting spinless bosons submitted to a rotation around the x^3 axis and a harmonic trapping potential. Denoting $\vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$, the Hamiltonian of the system in the rotating frame is given by

$$\sum_{j=1}^{N} \left[\frac{|\vec{p}_j|^2}{2m} + \frac{m}{2} \left(\omega_\perp^2 \left(|x_j^1|^2 + |x_j^2|^2 \right) + \omega_\parallel^2 |x_j^3|^2 \right) - \Omega \vec{e}_3 \cdot \vec{L}_j \right] + \sum_{1 \leq j < k \leq N} W_a(\vec{x}_j - \vec{x}_k). \quad (1)$$

Here $\vec{L} = \vec{x} \times \vec{p}$ is the angular momentum, $\vec{e}_3 = (0, 0, 1)$, m is the mass of the bosons, and $\omega_\perp$ and $\omega_\parallel$ are the trap frequencies. The interaction potential W_a is assumed to be non-negative, i.e., purely repulsive, and to have scattering length a (see [24] for a proper definition of the scattering length). It is natural to introduce a fixed potential W with scattering length 1 and write $W_a(x) = a^{-2} W(x/a)$. For convenience we will assume that the angular velocity Ω is nonnegative. The above Hamiltonian acts on the space of permutation-symmetric square-integrable N-body wavefunctions.

The Hamiltonian (1) is stable (i.e., bounded from below) only when $\Omega \leq \omega_\perp$. The regime of rapid rotation that will be of special interest to us corresponds to the case when Ω is very close to the maximal possible speed $\omega_\perp$, i.e.,

$$\omega := \frac{\omega_\perp - \Omega}{\omega_\perp} \ll 1.$$

To simplify certain expressions, we will work with an isotropic harmonic potential, $\omega_\perp = \omega_\parallel$. Our results would hold equally well when $\omega_\perp \neq \omega_\parallel$ but $\omega_\parallel/\omega_\perp \geq \varepsilon > 0$. Similarly, a non-harmonic confinement potential in the x^3 direction could be used.

It is convenient to chose units such that $m = \hbar = \omega_\perp = 1$. Introducing $\vec{A}(\vec{x}) = (-x^2, x^1, 0)$ and completing the square, our Hamiltonian (1) can be written as

$$H_{\omega,a}^N := \sum_{j=1}^{N} \left[\frac{|\vec{p}_j - \vec{A}(\vec{x}_j)|^2 + |x_j^3|^2 - 3}{2} + \omega \vec{e}_3 \cdot \vec{L}_j \right] + \sum_{1 \leq j < k \leq N} W_a(\vec{x}_j - \vec{x}_k). \quad (2)$$

The kinetic energy term of this Hamiltonian is equivalent to that of a charged particle in a constant magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. The spectrum of $(|\vec{p} - \vec{A}(\vec{x})|^2 + |x^3|^2)/2$ is purely discrete, its eigenvalues being $3(j + 1/2)$ for $j = 0, 1, \dots$. They are all infinitely degenerate. In the definition (2) of our Hamiltonian we have subtracted the unimportant ground state energy $3/2$ of the kinetic term.

When the speed of rotation Ω is not too close to $\omega_\perp$ and the Bose gas is sufficiently dilute, the ground state of (2) is known to exhibit Bose-Einstein condensation, with condensate wavefunction described by ground states of the Gross-Pitaevskii functional [18, 33]

$$\mathcal{E}^{\text{GP}}(\varphi) = \left\langle \varphi, \left(\frac{|\vec{p} - \vec{A}(\vec{x})|^2 + |x^3|^2 - 3}{2} + \omega \vec{e}_3 \cdot \vec{L} \right) \varphi \right\rangle + \frac{g}{2} \int_{\mathbb{R}^3} |\varphi|^4, \quad (3)$$

where $g = 4\pi Na$. In the limit $N \to \infty$ with $0 < \omega \leq 1$ and $g > 0$ fixed this was proved in [21, 22].

The properties of the Gross-Pitaevskii ground state in the rapidly rotating regime $\omega \to 0$ were intensely studied in the literature, both from a numerical [3, 9, 12] and an analytical [1, 5] point of view. As the speed of rotation increases, more and more vortices appear and the wavefunction acquires a higher angular momentum. The location of these vortices is conveniently studied in the *Lowest Landau Level* (LLL) approximation where one restricts φ to lie in the null space of $|\vec{p} - \vec{A}(\vec{x})|^2 + |x^3|^2 - 3$. This LLL approximation is justified [2] when $\omega \ll 1$ and $g\omega \ll 1$, the number of vortices being then proportional to

$$N_v \sim \sqrt{\frac{g}{\omega}} \sim \sqrt{\frac{Na}{\omega}}.$$

The Gross-Pitaevskii functional (3) is expected to be an accurate description of the ground state of the many-body system (2) provided the number of vortices is much smaller than the number of particles in the system, i.e., when

$$\frac{a}{N\omega} \ll 1. \tag{4}$$

Within the LLL approximation, this was recently shown in [27] to be indeed the case. In terms of the *filling factor* $\nu = N^2/(2L_{\text{tot}})$ [14], (4) corresponds to $\nu \gg 1$.

When $a/(N\omega)$ is not small, a completely different regime is expected. Evidence of strongly correlated states was found in exact diagonalization studies of small systems. As the rotation frequency Ω is increased from 0 to its upper limit $\omega_\perp$, the ground state encounters a series of transitions between certain values of the angular momentum. The behavior of the system is similar to the *Fractional Quantum Hall Effect* in fermionic systems, and usually modeled by an effective Hamiltonian with contact interaction in the LLL. The rigorous derivation of this effective Hamiltonian for Bose gases with generic repulsive two-body interactions is the main purpose of this paper.

2 Main Results

2.1 Derivation of the Effective Hamiltonian on the LLL

The ground state energy in the bosonic sector is given by

$$E_N(\omega, a) := \inf \sigma_{\bigvee_1^N L^2(\mathbb{R}^3, \mathbb{C})} \left(H_{\omega,a}^N \right) \tag{5}$$

where $\bigvee_1^N L^2(\mathbb{R}^3, \mathbb{C})$ denotes the *symmetric* tensor product and $H_{\omega,a}^N$ was defined above in (2).

We will compare the ground state energy of the above Hamiltonian (2) with the simplified model consisting in restricting the wavefunction to the N-body *Lowest Landau Level* (LLL) and replacing W_a by $4\pi a$ times a contact interaction potential. The LLL is defined as the ground state eigenspace of the kinetic part of the operator (2) at $\omega = 0$. This subspace of $\bigvee_1^N L^2(\mathbb{R}^3, \mathbb{C})$ reads

$$\mathfrak{H}_N := \left\{ \Psi(\vec{x}_1, \ldots, \vec{x}_N) = F(x_1^1 + ix_1^2, \ldots, x_N^1 + ix_N^2)e^{-\sum_{i=1}^{N} \frac{|\vec{x}_i|^2}{2}} \in L^2(\mathbb{R}^{3N}) : \right.$$

$$\left. (z_1, \ldots, z_N) \mapsto F(z_1, \ldots, z_N) \text{ is holomorphic and symmetric} \right\} \qquad (6)$$

where we denote $\vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$ as before. We will use the notation $z = x^1 + ix^2 \in \mathbb{C}$ and we will sometimes identify it with $(x^1, x^2) \in \mathbb{R}^2$. For any $\Psi \in \mathfrak{H}_N$, we have by definition

$$\sum_{j=1}^{N} \left[\frac{|\vec{p}_j - \vec{A}(\vec{x}_j)|^2 + |x_j^3|^2 - 3}{2} \right] \Psi = 0.$$

The ground state energy of the simplified effective model in the LLL is given by

$$E_N^{\text{LLL}}(\omega, a) := \inf_{\Psi \in \mathfrak{H}_N, \|\Psi\|=1} \left\langle \Psi, \left(\omega \sum_{j=1}^{N} \vec{e}_3 \cdot \vec{L}_j + 4\pi a \sum_{1 \leq i < j \leq N} \delta(\vec{x}_i - \vec{x}_j) \right) \Psi \right\rangle. \qquad (7)$$

Note that although it makes no sense to use a delta potential in the original Hilbert space, functions in the space $\mathfrak{H}_N$ are all smooth, hence

$$\langle \Psi, \delta(\vec{x}_1 - \vec{x}_2) \Psi \rangle = \int \cdots \int |\Psi(\vec{x}_2, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_N)|^2 dx_2 \cdots dx_N$$

makes perfect sense and defines a bounded selfadjoint operator. As we will discuss in the next section, for any $\omega \geq 0$ and $a \geq 0$ there exists a ground state $\Psi \in \mathfrak{H}_N$ for the problem (7).

We emphasize that (7) is *not* obtained by restricting $H_{\omega,a}^N$ to the LLL. For small scattering length a such a restriction would lead to a similar expression but with the wrong prefactor $\int W_a$ instead of $4\pi a$ in front of the δ-interaction (see Remark 1 below). In order to obtain the scattering length, it is important to note that the LLL restriction is unphysical on length scales much smaller than the effective "magnetic length", which is 1 in our units. If $a \ll 1$, the scattering process is unaffected by the rotation of the system and hence leads to the scattering length as an effective coupling constant.

Our main result is the following.

Theorem 1 (Validity of effective LLL model) *Let W be a nonnegative radial function such that $\int_{|\vec{x}|>R} W(\vec{x}) \, dx < \infty$ for some $R > 0$, with scattering length 1. We define $W_a := a^{-2} W(\cdot/a)$ and*

$$\kappa := \frac{a}{N\omega}. \qquad (8)$$

(i) ***Upper bound***. *Assume that $\eta := \kappa^{1/4} a N^{1/2} \leq 1$. For $\kappa^{-3/2} a < C^{-1}$ one has*

$$E_N(\omega, a) \leq E_N^{\text{LLL}}(\omega, a) \left(1 - C a \kappa^{-3/2} \right)^{-1}$$

$$\times \left(1 + \frac{C\eta}{\min\{1, \kappa N^2\}} \left[1 + \frac{\kappa^{-3/4}}{\sqrt{N}} + \frac{1}{\eta} \int_{|\vec{x}| \geq (2/\eta)^{3/4}} W \right] \right) \qquad (9)$$

for some universal constant $C > 0$.

(ii) **Lower bound**. *Let* $\xi = \min\{1, \kappa a^{-2/3}\}$. *Then*

$$E_N(\omega, a) \geq E_N^{\text{LLL}}(\omega, a) \left(1 - C\left[\frac{Na^{1/3}}{\xi} + a^{1/9}\xi\right] - \frac{1}{4\pi} \int_{|\vec{x}| \geq \xi^{1/6}a^{-8/9}} W\right) \quad (10)$$

for some universal constant $C > 0$.

What Theorem 1 says is that when κ stays away from zero and when a is small enough (depending on the particle number N), then one can replace the problem of minimizing $H_{\omega,a}^N$ with a generic interaction of scattering length a by the study of a simplified Hamiltonian acting on the LLL, with a contact interaction of strength $4\pi a$. The latter model has some very specific features which we will recall in the next section. There we shall also see that κ is related to the inverse filling factor, hence the regime $\kappa = O(1)$ is of particular relevance. Note that $a \to 0$ for fixed κ implies, in particular, that $\omega \to 0$.

A simple corollary of Theorem 1 is that for fixed N and fixed κ,

$$\lim_{a \to 0} \frac{E_N(\omega = a/(N\kappa), a)}{E_N^{\text{LLL}}(\omega = a/(N\kappa), a)} = 1.$$

We conjecture that this convergence is in fact uniform in N, but Theorem 1 does not show this. Although N is allowed to go to infinity as a goes to zero, our error bounds are only small if it does not increase too fast.

For fixed κ, the leading order correction in our upper bound is of the order $aN^{1/2}$ as long as W decays at least as $|\vec{x}|^{-3-4/3}$ at infinity; it is $(aN^{1/2})^{3\varepsilon/4}$ if W decays as $|\vec{x}|^{-3-\varepsilon}$ for $0 < \varepsilon < 4/3$ instead. This follows from the fact that, for small η, $\frac{1}{\eta} \int_{|\vec{x}| \geq (2/\eta)^{3/4}} W \sim \eta^{3\varepsilon/4-1}$ if W decays as $|\vec{x}|^{-3-\varepsilon}$ for $\varepsilon > 0$.

The error bounds in the lower bound are significantly worse. For fixed κ the leading error term is $Na^{1/3}$. It remains a challenging open problem to derive bounds that display a better N dependence. These will require a better understanding of the FQHE regime for large N.

The proof of Theorem 1 uses several previous ideas [24]. The upper bound requires a two-scale trial function, as suggested first by Dyson in [16], in order to obtain the scattering length from W_a. The fact that the ground state in $\mathfrak{H}_N$ for (7) is not very well known (contrarily to the condensed Gross-Pitaevskii case) is an important obstacle, however. As usual, the lower bound is the hardest part and consequently our conditions on a are more restrictive.

The proof of Theorem 1 will be given in Sect. 3.2. More general upper and lower bounds on $E_N(\omega, a)$ are stated in (28)–(29) and (43), respectively.

2.2 Effective Hamiltonian on the LLL and Convergence of States

The result of Theorem 1 can be extended to obtain information not only on the ground state energy but also on the corresponding eigenfunctions. Before we state our result on the convergence of ground states in the limit $\omega \to 0$ and $a \to 0$ with N fixed, we recall in this section several important properties of the effective Hamiltonian on the LLL.

It is convenient to introduce the *Bargmann space* [7]

$$\mathcal{B}_N := \left\{ F : \mathbb{C}^N \to \mathbb{C}^N \text{ holomorphic and symmetric} : \right.$$

$$\left. \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} |F(z_1, \ldots, z_N)|^2 e^{-\sum_{j=1}^N |z_j|^2} dz_1 \cdots dz_N < \infty \right\}$$

endowed with the scalar product

$$\langle F, G \rangle_{\mathcal{B}_N} := \int_{\mathbb{C}} dz_1 \cdots \int_{\mathbb{C}} dz_N \overline{F(z_1, \ldots, z_N)} G(z_1, \ldots, z_N) e^{-\sum_{j=1}^{N} |z_j|^2},$$

and its associated norm. It can easily be checked that if $F \in \mathcal{B}_N$ then the function Ψ defined by $\Psi = \pi^{-N/4} e^{-\sum_{j=1}^{N} |\vec{x}_j|^2/2} F \in \mathfrak{H}_N$ satisfies $\|\Psi\|_{L^2(\mathbb{R}^{3N})} = \|F\|_{\mathcal{B}_N}$ and

$$\sum_{j=1}^{N} \left[\frac{|\vec{p}_j - \vec{A}(\vec{x}_j)|^2 + |x_j^3|^2 - 3}{2} + \omega \vec{e}_3 \cdot \vec{L}_j \right] \Psi = \omega \left(\sum_{j=1}^{N} z_j \partial_{z_j} F \right) e^{-\sum_{i=1}^{N} \frac{|\vec{x}_i|^2}{2}}. \quad (11)$$

The delta interaction potential is defined on $\mathcal{B}_N$ as follows:

$$\left(\sum_{i<j} \delta_{ij} \right) F := \frac{1}{(2\pi)^{3/2}} \sum_{i<j} F \left(z_1, \ldots, \frac{z_i + z_j}{2}, \ldots, \frac{z_i + z_j}{2}, \ldots, z_N \right). \quad (12)$$

The prefactor has been chosen to ensure that

$$\langle F, \delta_{12} F \rangle_{\mathcal{B}_2} = \int_{\mathbb{R}^3} |\Psi(\vec{x}, \vec{x})|^2 dx.$$

It can easily be seen that $0 \le \delta_{12} \le (2\pi)^{-3/2}$, hence δ_{12} is a bounded self-adjoint operator on $\mathcal{B}_2$. The model Hamiltonian acting on $\mathcal{B}_N$ is defined as

$$\tilde{H}_{\omega,a}^N := \omega \sum_{j=1}^{N} z_j \partial_{z_j} + 4\pi a \sum_{1 \le i < j \le N} \delta_{ij} \quad (13)$$

and its ground state energy equals the LLL energy (7) introduced in the previous section:

$$E_N^{\mathrm{LLL}}(\omega, a) := \inf \sigma_{\mathcal{B}_N}(\tilde{H}_{\omega,a}^N).$$

We introduce, for convenience, the notation

$$\mathcal{L}_N := \sum_{j=1}^{N} z_j \partial_{z_j} \quad \text{and} \quad \mathcal{I}_N := \sum_{1 \le i < j \le N} \delta_{ij}$$

for the total angular momentum and the contact interaction potential in the LLL, respectively. Because of rotation invariance of the interaction these two operators commute on $\mathcal{B}_N$, i.e., $[\mathcal{L}_N, \mathcal{I}_N] = 0$. Hence the ground state energy $E_N^{\mathrm{LLL}}(\omega, a)$ of our Hamiltonian $\tilde{H}_{\omega,a}^N = \omega \mathcal{L}_N + 4\pi a \mathcal{I}_N$ is obtained by looking at the *joint spectrum* of $\mathcal{I}_N$ and $\mathcal{L}_N$. If we denote by $I_N(L)$ the lowest eigenvalue of the operator $\mathcal{I}_N$ in the sector of total angular momentum L, we get

$$E_N^{\mathrm{LLL}}(\omega, a) = \inf_{L \in \mathbb{N}} \{\omega L + 4\pi a I_N(L)\}.$$

Multiplying any common eigenstate of $\mathcal{L}_N$ and $\mathcal{I}_N$ by the center of mass $\sum_{j=1}^{N} z_j$, one sees that $\sigma(\mathcal{I}_N)_{\restriction(\mathcal{L}_N = L)} \subset \sigma(\mathcal{I}_N)_{\restriction(\mathcal{L}_N = L+1)}$. Therefore, $L \mapsto I_N(L)$ is nonincreasing.
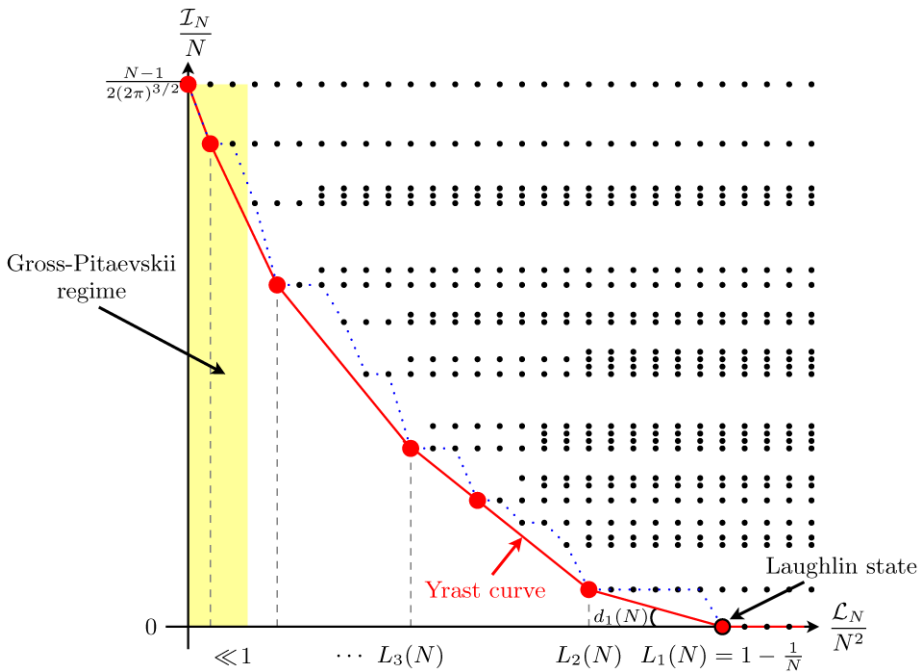
Fig. 1 General form of the joint spectrum of $\mathcal{I}_N/N$ and $\mathcal{L}_N/N^2$. The *dashed curve* is the graph of $\ell \mapsto I_N(\ell N^2)/N$, whereas the *solid one* is the *yrast curve*. Points of the joint spectrum lying on the yrast curve are emphasized by *thick dots*. This figure represents only a sketch, numerical studies for the joint spectrum of $\mathcal{I}_N$ and $\mathcal{L}_N$ can be found in [6, 35, 40]

A sketch of the general form of the joint spectrum of $\mathcal{I}_N$ and $\mathcal{L}_N$ is shown in Fig. 1. The possible ground states for $\tilde{H}_{\omega,a}^N$ are those whose values of $\mathcal{L}_N$ and $\mathcal{I}_N$ lie on the so-called *yrast curve*[1] [31] which is the graph of the convex hull of $L \mapsto I_N(L)$. We can write

$$\tilde{H}_{\omega,a}^N = 4\pi N a \left(\frac{1}{4\pi\kappa} \frac{\mathcal{L}_N}{N^2} + \frac{\mathcal{I}_N}{N} \right)$$

where, as before $\kappa = a/(N\omega)$. Thus the ground state that will be picked by the system only depends on the value of κ. It jumps from one state to another when κ is varied. The FQHE regime corresponds to $\kappa \sim 1$ in which case $\mathcal{L}_N \sim N^2$ and $\mathcal{I}_N \sim N$, hence $E_N^{\mathrm{LLL}}(\omega, a) \sim Na$.

The null space of $\mathcal{I}_N$ is obviously given by the set of functions

$$\left\{ F(z_1, \ldots, z_N) \prod_{1 \le i < j \le N} (z_i - z_j)^2 \middle| F \text{ holomorphic and symmetric} \right\}.$$

---

[1]In the literature the graph of $L \mapsto I_N(L)$ is sometimes called the yrast curve. We keep this name for the convex hull which contains all the possible ground states of $\tilde{H}_{\omega,a}^N$.

Fig. 2 Value of the angular momentum of the ground state(s) of $\tilde{H}^N_{\omega,a}$, depending on the value of κ. The Laughlin state is the unique ground state for all $\kappa > \kappa_1(N)$. The constant function is the unique ground state for all $0 \leq \kappa < \kappa_{j_{max}}(N)$
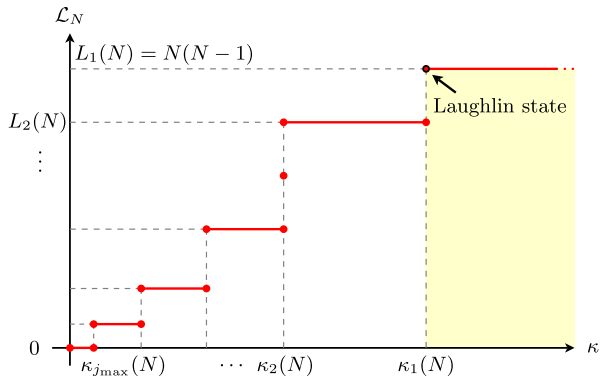
The function which has the lowest angular momentum among these functions (hence lies on the yrast curve) is the (bosonic) *Laughlin wavefunction*

$$F^N_{Lau}(z_1, \ldots, z_N) = k_N \prod_{1 \leq i < j \leq N} (z_i - z_j)^2$$

where k_N is a normalization factor. It satisfies

$$\mathcal{L}_N F^N_{Lau} = N(N-1) F^N_{Lau}.$$

The Laughlin wavefunction is the unique ground state of $\tilde{H}^N_{\omega,a}$ as soon as $\kappa > \kappa_1(N) := -1/(4\pi d_1(N))$, where $d_1(N)$ is the (unknown) left derivative at $\ell = 1 - 1/N$ of the convex hull of $\ell \mapsto I_N(N^2\ell)/N$. The *filling factor* [14] of the Laughlin function is given by

$$\nu_{Lau} = \frac{N^2}{2\langle F^N_{Lau}, \mathcal{L}_N F^N_{Lau}\rangle} = \frac{1}{2(1-1/N)} \xrightarrow{N \to \infty} \frac{1}{2}.$$

In general, the convex hull $\tilde{I}_N$ of the function $L \mapsto I_N(L)$ is piecewise linear and we may define similarly $\kappa_1(N) > \kappa_2(N) > \cdots > \kappa_k(N)$ by the formula $\kappa_j(N) := -1/(4\pi d_j(N))$ where $d_1(N) > d_2(N) > \cdots > d_k(N)$ are the successive left derivatives of the function $\ell \mapsto \tilde{I}_N(N^2\ell)/N$. To any $\kappa_j(N)$ we can associate a unique total angular momentum $L_j(N)$ which is the highest among states lying on the yrast curve and having a left derivative equal to $d_j(N)$. The corresponding eigenspace is easily seen to be the ground state eigenspace of $\tilde{H}^N_{\omega,a}$ when $\kappa \in (\kappa_{j+1}(N), \kappa_j(N))$. When $\kappa = \kappa_j(N)$, the ground state eigenspace is the one containing all states lying on the yrast line with slope $d_j(N)$. It does not have a unique angular momentum. These statements are illustrated in Fig. 2.

The only state having $L = 0$ is the condensed state $F(z_1, \ldots, z_N) = 1$, hence $I_N(0) = (2\pi)^{-3/2} N(N-1)/2$. Also $I_N(1) = (2\pi)^{-3/2} N(N-1)/2$, with unique state $F(z_1, \ldots, z_N) = \sum_{i=1}^N z_i$. Moreover, it is well known [8, 19, 32] that

$$I_N(L) = \frac{1}{2(2\pi)^{3/2}} N\left(N - 1 - \frac{1}{2}L\right) \quad \text{for } 2 \leq L \leq N. \tag{14}$$

For the proof, one notes that δ_{12} has eigenvalues 0 and $(2\pi)^{-3/2}$ and commutes with the relative angular momentum $L_{12} = (z_1 - z_2)(\partial_{z_1} - \partial_{z_2})/2$; in fact, δ_{12} is nonzero only on

the subspace where $L_{12} = 0$. On *symmetric* functions of z_1 and z_2, the smallest non-zero eigenvalue of L_{12} is 2, hence $(2\pi)^{3/2}\delta_{12} \geq 1 - L_{12}/2$. Summing over all pairs we get

$$(2\pi)^{3/2} \sum_{i<j} \delta_{ij} \geq \frac{N(N-1)}{2} - \frac{N\mathcal{L}_N}{4} + \frac{1}{4}\left(\sum_i z_i\right)\left(\sum_i \partial_{z_i}\right).$$

The very last term is non-negative, which yields (14) as a lower bound. Finally, one checks that for $2 \leq L \leq N$ the lower bound is, in fact, an equality for the states $\mathcal{S}(z_1 - z_{CM}) \cdots (z_L - z_{CM})$, where $\mathcal{S}$ denotes symmetrization and $z_{CM} := N^{-1}\sum_{i=1}^N z_i$.

No exact formula for $I_N(L)$ is known if $L > N$. For large N and $L \ll N^2$ the yrast line was studied in [27], where it is proved that in this limit the Gross-Pitaevskii energy becomes exact. The result in [27] implies that in this regime the convex hull of $I_N(L)$ is proportional to N^3/L.

Very little is known about the yrast curve for $L \sim N^2$, in particular concerning lower bounds. Upper bounds have been derived using certain trial states (Pfaffian, composite fermions [13, 34, 35]) which have been shown numerically to have a large overlap with (some of) the true eigenstates of the yrast curve, at least for small N. A rigorous understanding of the properties of the true eigenstates is still missing, however. It particular, it remains an open problem to investigate whether $\liminf_{N\to\infty} d_1(N) > 0$. This would imply that the yrast curve has a discontinuous derivative at the Laughlin state. It would also show a certain robustness of the Laughlin state, in the sense that this state is the ground state for fixed $\kappa > \kappa_1 := \limsup_{N\to\infty} \kappa_1(N)$, independently of the particle number N.

This concludes our review of the properties of the effective Hamiltonian (13). To state our last result, we will denote by $P_N(\kappa)$ the (finite dimensional) orthogonal projector in $L^2(\mathbb{R}^{3N})$ on the ground eigenspace of the operator $(4\pi\kappa)^{-1}\mathcal{L}_N/N^2 + \mathcal{I}_N/N$, multiplied by $\pi^{-N/4}e^{-\sum_{j=1}^N |\vec{x}_j|^2/2}$. This orthogonal projector is constant for all $\kappa \in (\kappa_{j+1}(N), \kappa_j(N))$. For $\kappa > \kappa_1(N)$, it is just the projector on the N-body Laughlin function

$$P_N(\kappa) = |\Psi_{Lau}^N\rangle\langle\Psi_{Lau}^N|$$

where

$$\Psi_{Lau}^N(\vec{x}_1, \ldots, \vec{x}_N) = k_N\pi^{-N/4}\prod_{1\leq i<j\leq N}(z_i - z_j)^2 e^{-\sum_{k=1}^N |\vec{x}_k|^2/2}.$$

The following theorem is a rather straightforward consequence of our proof of Theorem 1.

Theorem 2 (Convergence of States and Fractional Quantum Hall Effect) *Let $\kappa > 0$ and $N \geq 2$ fixed, and denote by $\Psi_{\omega,a}^N$ any chosen sequence of ground states of the Hamiltonian $H_{\omega,a}^N$ in $\bigvee_1^N L^2(\mathbb{R}^3)$. Then we have*

$$\lim_{\substack{a\to 0 \\ a/(N\omega)\to\kappa}} \left\|\Psi_{\omega,a}^N - P_N(\kappa)\Psi_{\omega,a}^N\right\| = 0. \tag{15}$$

In particular, if $\kappa > \kappa_1(N)$, one has $\Psi_{\omega,a}^N \to \Psi_{Lau}^N$ when $a \to 0$ and $a/(N\omega) \to \kappa$, up to a correct choice of a phase for $\Psi_{\omega,a}^N$.

Theorem 2 shows that the ground state of any system of N spinless trapped bosons with repulsive interactions of scattering length a is, for small enough a and rotation speed close to the critical one, well approximated by the ground state of the effective Hamiltonian (13)

on the LLL. In particular, the states are highly correlated and exhibit a bosonic analogue of the FQHE, with transitions between certain values of the angular momentum coinciding with discontinuities of the derivative of the yrast curve.

The proof of Theorem 2 will be given in Sect. 3.3. As will be obvious from the method of proof, both Theorems 1 and 2 can be extended to low-lying excited energy eigenvalues and their corresponding eigenfunctions as well. The corresponding analysis is similar to previous studies of the effective one-dimensional behavior of Bose gases in highly elongated traps [39], and we shall not give the details here.

3 Proofs

3.1 Preliminaries

In this subsection we shall gather some useful preliminary results which will be needed in the rest of the proof. We recall that δ_{ij} was defined in (12). Similarly the three-body delta interaction can be defined on $\mathcal{B}_3$ as

$$(\delta_{123}F)(z_1, z_2, z_3) := \frac{1}{(\sqrt{3}\pi)^3} F\left(\frac{z_1 + z_2 + z_3}{3}, \frac{z_1 + z_2 + z_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right) \quad (16)$$

where the prefactor was chosen such that

$$\langle F, \delta_{123}F \rangle_{\mathcal{B}_3} = \left(\int_{\mathbb{R}} e^{-t^2} dt\right)^{-3} \int_{\mathbb{R}} dx \int_{\mathbb{C}} dz |F(z, z, z)|^2 e^{-3|z|^2 - 3x^2}.$$

It defines a bounded self-adjoint operator on $\mathcal{B}_3$. In fact,

$$\delta_{123} \leq \sqrt{\frac{2}{3\pi^3}} \delta_{12}. \quad (17)$$

By definition all functions in $\mathfrak{H}_N$ are smooth. A way to quantify their regularity was provided by Carlen in [10]. We state it in the following lemma.

Lemma 1 (An inequality of Carlen [10]) *For any $p \in \mathbb{N}$, there exists a constant C_p such that for any holomorphic function $f \in \mathcal{B}_1$*

$$\forall z \in \mathbb{C}, \quad \left|\frac{\partial^p f}{\partial z^p}(z)\right|^2 e^{-|z|^2} \leq C_p(1 + |z|^{2p}) \|f\|_{\mathcal{B}_1}^2. \quad (18)$$

Proof The proof is a consequence of the Cauchy-Schwarz inequality and the following well-known *coherent state representation* [5, 7, 17] for functions f in the Bargmann space $\mathcal{B}_1$:

$$f(z) = \pi^{-1} \int_{\mathbb{C}} f(\xi) \varphi_\xi(z) e^{-|\xi|^2} d\xi \quad (19)$$

where $\varphi_\xi(z) = e^{\bar{\xi}z}$. $\qquad \square$

Using the smoothness of functions in the LLL, one can control any interaction potential by a contact interaction, up to an error. This was done first in [4].

Lemma 2 (Controlling interaction potentials in the LLL) *Let $F \in \mathcal{B}_N$ and $\Psi(\vec{x}_1, \ldots, \vec{x}_N) :=$* $\pi^{-N/4} F(z_1, \ldots, z_N) e^{-\sum_{j=1}^N |\vec{x}_j|^2/2} \in \mathfrak{H}_N$. *Let $g \in L^1(\mathbb{R}^3)$ be non-negative and radial. Then we have*

$$
\left\langle \Psi, \left(\sum_{1 \leq i \neq j \leq N} g(\vec{x}_i - \vec{x}_j) \right) \Psi \right\rangle_{L^2(\mathbb{R}^{3N})}
$$

$$
\leq \left(\int_{\mathbb{R}^3} g \right) \left\langle F, \left(\sum_{1 \leq i \neq j \leq N} \delta_{ij} \right) F \right\rangle_{\mathcal{B}_N} + N^2 C \left(\int_{\mathbb{R}^3} g(\vec{x}) \frac{|\vec{x}|^4}{1 + |\vec{x}|^4} \, dx \right) \|F\|_{\mathcal{B}_N}^2, \quad (20)
$$

and

$$
\left\langle \Psi, \left(\sum_{1 \leq i \neq j \neq k \leq N} g(\vec{x}_i - \vec{x}_j) g(\vec{x}_j - \vec{x}_k) \right) \Psi \right\rangle_{L^2(\mathbb{R}^{3N})}
$$

$$
\leq \left(\int_{\mathbb{R}^3} g \right)^2 \left\langle F, \left(\sum_{1 \leq i \neq j \neq k \leq N} \delta_{ijk} \right) F \right\rangle_{\mathcal{B}_N} + N^3 C \left(\int_{\mathbb{R}^3} g(\vec{x}) \frac{|\vec{x}|^2}{1 + |\vec{x}|^2} dx \right)^2 \|F\|_{\mathcal{B}_N}^2 \quad (21)
$$

for a universal constant $C > 0$.

Proof An inequality similar to (20) was derived before in [4]. We shall only write the proof of (20) for $N = 2$ (the general case is then obtained by summing over pairs). We shall omit the proof of (21) which is analogous.

Defining $G(u, v) = F(\frac{u+v}{\sqrt{2}}, \frac{u-v}{\sqrt{2}})$, we have

$$
\langle \Psi, g(\vec{x}_1 - \vec{x}_2) \Psi \rangle = \frac{1}{\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g(\vec{x}_1 - \vec{x}_2) |F(z_1, z_2)|^2 e^{-|\vec{x}_1|^2 - |\vec{x}_2|^2} dx_1 \, dx_2
$$

$$
= \int_{\mathbb{C}} \int_{\mathbb{C}} \tilde{g}(\sqrt{2}|v|) |G(u, v)|^2 e^{-|u|^2 - |v|^2} du \, dv, \quad (22)
$$

where we have introduced

$$
\tilde{g}(|z|) := \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} g(z, x_1^3 - x_2^3) e^{-|x_1^3|^2 - |x_2^3|^2} dx_1^3 \, dx_2^3
$$

which is obviously radial, i.e., depends only on $|z|$.

We split the v integral in (22) into two parts, corresponding to $|v| \leq 1$ and $|v| \geq 1$, respectively. Consider first the case $|v| \leq 1$. Using the fact that $G(u, \cdot)$ is even because F is symmetric, a Taylor expansion yields

$$
G(u, v) = G(u, 0) + v^2 \int_0^1 (1 - t) \frac{\partial^2 G}{\partial v^2} (u, tv) \, dt.
$$

By the radiality of $\tilde{g}$ the cross term vanishes when integrating over angles, and hence

$$
\int_{|v| \leq 1} \tilde{g}(\sqrt{2}|v|) |G(u, v)|^2 e^{-|v|^2} dv
$$

$$= \int_{|z| \leq 1} \tilde{g}(\sqrt{2}|v|)|G(u,0)|^2 e^{-|v|^2} dv$$

$$+ \int_{|v| \leq 1} \tilde{g}(\sqrt{2}|v|)|v|^4 \left| \int_0^1 (1-t) \frac{\partial^2 G}{\partial v^2}(u,tv) \, dt \right|^2 e^{-|v|^2} dv. \qquad (23)$$

We integrate this identity against $e^{-|u|^2} du$. The first term becomes

$$\left(\int_{|z| \leq 1/\sqrt{2}} g(z, x^3) e^{-|\vec{x}|^2/2} dx \right) \langle F, \delta_{12} F \rangle_{\mathcal{B}_2} \leq \left(\int_{\mathbb{R}^3} g \right) \langle F, \delta_{12} F \rangle_{\mathcal{B}_2}.$$

With the aid of Carlen's inequality (18), the second term is bounded above by

$$C \int_{|z| \leq 1/\sqrt{2}} g(z, x^3) |z|^4 \, dx \, \|F\|_{\mathcal{B}_2}^2 \leq C' \int_{\mathbb{R}^3} g(\vec{x}) \frac{|\vec{x}|^4}{1 + |\vec{x}|^4} \, dx \, \|F\|_{\mathcal{B}_2}^2.$$

Finally, for $|v| \geq 1$ we shall use again (18), this time for $p = 0$, to conclude that

$$\int_{\mathbb{C}} du \int_{|v| \geq 1} dv \tilde{g}(\sqrt{2}|v|)|G(u,v)|^2 e^{-|u|^2-|v|^2} \leq C \int_{|z| \geq 1/\sqrt{2}} g(z, x^3) dx \, \|F\|_{\mathcal{B}_2}^2$$

$$\leq C' \int_{\mathbb{R}^3} g(\vec{x}) \frac{|\vec{x}|^4}{1 + |\vec{x}|^4} \, dx \, \|F\|_{\mathcal{B}_2}^2.$$

This completes the proof. □

Remark 1 Although we will not need it, we note that (23) also yields a lower bound:

$$\langle \Psi, g(\vec{x}_1 - \vec{x}_2)\Psi \rangle_{\mathfrak{H}_2} \geq \left(\int_{\mathbb{R}^3} g(\vec{x}) e^{-\frac{|\vec{x}|^2}{2}} \, dx \right) \langle F, \delta_{12} F \rangle_{\mathcal{B}_2},$$

where we have used the same notation as in Lemma 2. Combined with (20) this shows that the restriction of the operator $\varepsilon^{-3} g((\vec{x}_1 - \vec{x}_2)/\varepsilon)$ to the LLL converges to $(\int_{\mathbb{R}^3} g)\delta_{12}$ as $\varepsilon \to 0$.

It will be important to have some *a priori* bounds on the ground state energy of the effective Hamiltonian (13). The following is certainly not optimal but it has the merit of being simple.

Lemma 3 (Simple Bounds on $E_N^{\text{LLL}}(\omega, a)$) *We have, for $\kappa = a/(N\omega)$ and $N \geq 2$,*

$$aNC \min \left\{ \frac{1}{\kappa N}, N \right\} \leq E_N^{\text{LLL}}(\omega, a) \leq aN \min \left\{ \frac{1}{\kappa}, \sqrt{\frac{2}{\pi}} N \right\}. \qquad (24)$$

Proof The upper bound is obtained by taking as trial states the Laughlin function and the constant function, respectively. For the lower bound, we note that

$$\sum_{1 \leq i < j \leq N} \left(\frac{\omega}{N-1}(L_{z_i} + L_{z_j}) + 4\pi a \delta_{ij} \right)$$

$$\geq c \min \{\omega N/2, 2\pi a N(N-1)\} \geq \frac{c}{2} \min \{\kappa^{-1} a, aN^2\}$$

where $c = \inf \sigma_{\mathcal{B}_2}(L_1 + L_2 + \delta_{12}) > 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

Except for the prefactor, the upper bound is expected to be sharp. In other words, the lower bound should hold without the factor $1/N$ multiplying κ^{-1}, for an appropriate constant C. This remains an open problem, however.

3.2 Proof of Theorem 1

Step 1: Upper Bound   We start by proving the upper bound, using the variational principle. The main difficulty is to get the scattering length in front of the interaction potential. As suggested first by Dyson in [16], this is done by multiplying a trial state Φ of $\mathfrak{H}_N$ by a correlated function S accounting for the short scale structure of the ground state. Compared to previous similar arguments in [25, 37], a new complication comes from the fact that the trial state Φ in $\mathfrak{H}_N$ is not a simple product function, but itself already a (possibly) highly correlated state of which little is known. Fortunately, the information that $\Phi \in \mathfrak{H}_N$ combined with simple bounds on $E_N^{\mathrm{LLL}}(\omega, a)$ will allow us to get the desired upper bound.

Let $F_{N,\omega,a} \in \mathcal{B}_N$ be a normalized ground state for the LLL Hamiltonian $\tilde{H}_{\omega,a}^N$ defined in (13), which is a common eigenvector of $\mathcal{L}_N$ and $\mathcal{I}_N$. We consider the following trial state:

$$\Psi_{N,\omega,a} := S_{N,a} \Phi_{N,\omega,a} \tag{25}$$

where

$$S_{N,a}(\vec{x}_1, \ldots, \vec{x}_N) := \prod_{1 \le i < j \le N} f_a(|\vec{x}_i - \vec{x}_j|)$$

for some $0 \le f_a \le 1$ which will be defined later, and

$$\Phi_{N,\omega,a}(\vec{x}_1, \ldots, \vec{x}_N) := \pi^{-N/4} F_{N,\omega,a}(z_1, \ldots, z_N) e^{-\sum_{i=1}^N \frac{|\vec{x}_i|^2}{2}}.$$

Note that the norm of $\Phi_{N,\omega,a} \in L^2(\mathbb{R}^{3N})$ equals the norm of $F_{N,\omega,a} \in \mathcal{B}_N$. We write

$$H_{\omega,a}^N = \sum_{j=1}^N (h_\omega)_j + \sum_{i<j} W_a(\vec{x}_i - \vec{x}_j)$$

where

$$h_\omega = \frac{|\vec{p} - \vec{e}_3 \times \vec{x}|^2}{2} + \frac{|x^3|^2}{2} - \frac{3}{2} + \omega \vec{e}_3 \cdot \vec{L}. \tag{26}$$

We shall also use the notation $H_{\omega,0}^N = \sum_j (h_\omega)_j$ for short. Using the fact that f_a is real, we can argue as in [37, (4.64)] to get the identity

$$\langle \Psi_{N,\omega,a}, H_{\omega,0}^N \Psi_{N,\omega,a} \rangle = \sum_{j=1}^N \int \frac{|\vec{\nabla}_j S_{N,a}|^2}{2} |\Phi_{N,\omega,a}|^2 + \mathrm{Re} \langle S_{N,a}^2 \Phi_{N,\omega,a}, H_{\omega,0}^N \Phi_{N,\omega,a} \rangle.$$

Using (11) and that $F_{N,\omega,a}$ is a normalized eigenvector of $\mathcal{L}_N$, we have

$$H_{\omega,0}^N \Phi_{N,\omega,a} = \omega \langle F_{N,\omega,a}, \mathcal{L}_N F_{N,\omega,a} \rangle_{\mathcal{B}_N} \Phi_{N,\omega,a},$$

hence

$$\langle \Psi_{N,\omega,a}, H^N_{\omega,0} \Psi_{N,\omega,a} \rangle = \sum_{j=1}^{N} \int \frac{|\vec{\nabla}_j S_{N,a}|^2}{2} |\Phi_{N,\omega,a}|^2$$

$$+ \left\| \Psi_{N,\omega,a} \right\|^2 \langle \Phi_{N,\omega,a}, H^N_{\omega,0} \Phi_{N,\omega,a} \rangle.$$

We compute

$$\vec{\nabla}_k S_{N,a} = \sum_{i \neq k} f'_a(|\vec{x}_k - \vec{x}_i|) \frac{\vec{x}_k - \vec{x}_i}{|\vec{x}_k - \vec{x}_i|} \prod_{\substack{1 \leq m < n \leq N \\ \{m,n\} \neq \{i,k\}}} f(|\vec{x}_m - \vec{x}_n|).$$

Using $0 \leq f \leq 1$ we therefore get

$$\frac{1}{2} \sum_{k=1}^{N} |\vec{\nabla}_k S_{N,a}|^2 \leq \sum_{1 \leq i < j \leq N} f'_a(|\vec{x}_i - \vec{x}_j|)^2 + \frac{1}{2} \sum_{i \neq j \neq k} f'_a(|\vec{x}_i - \vec{x}_j|) f'_a(|\vec{x}_k - \vec{x}_j|).$$

We finally deduce that

$$\langle \Psi_{N,\omega,a}, H^N_{\omega,a} \Psi_{N,\omega,a} \rangle \leq \left\| \Psi_{N,\omega,a} \right\|^2 \langle \Phi_{N,\omega,a}, H^N_{\omega,0} \Phi_{N,\omega,a} \rangle$$

$$+ \left\langle \Phi_{N,\omega,N,a}, \left(\sum_{1 \leq i < j \leq N} [(f'_a)^2 + W_a f_a^2](|\vec{x}_i - \vec{x}_j|) \right) \Phi_{N,\omega,N,a} \right\rangle$$

$$+ \frac{1}{2} \left\langle \Phi_{N,\omega,N,a}, \left(\sum_{i \neq j \neq k} f'_a(|\vec{x}_i - \vec{x}_j|) f'_a(|\vec{x}_k - \vec{x}_j|) \right) \Phi_{N,\omega,N,a} \right\rangle.$$

The next step is to bound the terms on the right hand side of the previous inequality. An essential tool is the inequality (20) of Lemma 2 which relates, on the LLL, the interaction of a smooth potential $\sum_{i<j} g(\vec{x}_i - \vec{x}_j)$ with that of the contact interaction with coefficient $\int_{\mathbb{R}^3} g$. As we will see, for a correct choice of f_a, we will have $\int [(f'_a)^2 + W_a f_a^2] \simeq 4\pi a$, as desired. Let $g_a := (f'_a)^2 + W_a(f_a)^2$. Using Lemma 2 as well as the bound (17), we obtain

$$\langle \Psi_{N,\omega,a}, H^N_{\omega,a} \Psi_{N,\omega,a} \rangle$$

$$\leq E_N^{\mathrm{LLL}}(\omega, a) \left\| \Psi_{N,\omega,a} \right\|^2$$

$$+ \left(\int_{\mathbb{R}^3} g_a - 4\pi a \left\| \Psi_{N,\omega,a} \right\|^2 + \sqrt{\frac{2}{3\pi^3}} N \left(\int f'_a \right)^2 \right) \langle F_{N,\omega,a}, \mathcal{I}_N F_{N,\omega,a} \rangle_{\mathcal{B}_N}$$

$$+ C N^2 \int_{\mathbb{R}^3} g_a(\vec{x}) \frac{|\vec{x}|^4}{1 + |\vec{x}|^4} dx + C N^3 \left(\int_{\mathbb{R}^3} f'_a(|\vec{x}|) \frac{|\vec{x}|^2}{1 + |\vec{x}|^2} dx \right)^2.$$

For an upper bound, we can use $\langle F_{N,\omega,a}, \mathcal{I}_N F_{N,\omega,a} \rangle_{\mathcal{B}_N} \leq E_N^{\mathrm{LLL}}(\omega, a)/(4\pi a)$, hence

$$\langle \Psi_{N,\omega,a}, H^N_{\omega,a} \Psi_{N,\omega,a} \rangle \leq \frac{E_N^{\mathrm{LLL}}(\omega, a)}{4\pi a} \left(\int_{\mathbb{R}^3} g_a + \sqrt{\frac{2}{3\pi^3}} N \left(\int f'_a \right)^2 \right)$$

$$+ CN^2 \int_{\mathbb{R}^3} g_a(\vec{x}) \frac{|\vec{x}|^4}{1 + |\vec{x}|^4} dx + CN^3 \left(\int_{\mathbb{R}^3} f_a'(|\vec{x}|) \frac{|\vec{x}|^2}{1 + |\vec{x}|^2} dx \right)^2.$$

Let us now choose f_a. As in [24, 25] we take, for some $b > a$ to be specified later,

$$f_a(s) := \begin{cases} \frac{u_a(s)/s}{u_a(b)/b} & \text{if } 0 \le s \le b, \\ 1 & \text{if } s \ge b \end{cases} \tag{27}$$

where u_a is the solution of the scattering equation

$$-u_a''(s) + W_a(s)u_a(s) = 0$$

with $u_a(0) = 0$ and $\lim_{s \to \infty} u_a'(s) = 1$. Integrating by parts and using that $0 \le u_a(s) \le s$ and $0 \le s u_a'(s) - u_a(s) \le a$ [24, 25] we see that

$$\int_{\mathbb{R}^3} g_a = \int_{\mathbb{R}^3} [(f_a')^2 + W_a f_a^2] \le \frac{4\pi a}{1 - a/b}.$$

By splitting the integral into a part $|\vec{x}| \le (ab^3)^{1/4}$ and $|\vec{x}| \ge (ab^3)^{1/4}$, one checks that

$$\int_{\mathbb{R}^3} g_a(|\vec{x}|) \frac{|\vec{x}|^4}{1 + |\vec{x}|^4} dx \le \frac{C}{1 - a/b} \left(a^2 b^3 + ab^4 \int_{|\vec{x}| \ge (b/a)^{3/4}} W(\vec{x}) dx \right).$$

Note that for $a \ll b$ the second term in the last bracket is small compared to the first one if W decays faster than $|\vec{x}|^{-3-4/3}$ at infinity. We further have

$$\int_{\mathbb{R}^3} f_a' \le \frac{4\pi ab}{1 - a/b},$$

and hence

$$\int_{\mathbb{R}^3} f_a'(|\vec{x}|) \frac{|\vec{x}|^2}{1 + |\vec{x}|^2} dx \le \frac{4\pi ab^3}{1 - a/b}.$$

Let us assume, for simplicity, that $b \ge 2a$. Then

$$\left\langle \Psi_{N,\omega,a}, H_{\omega,a}^N \Psi_{N,\omega,a} \right\rangle \le E_N^{\text{LLL}}(\omega, a) \left(1 + C \left[\frac{a}{b} + Nab^2 \right] \right)$$

$$+ CN^2 a^2 b^3 \left(1 + Nb^3 + \frac{b}{a} \int_{|\vec{x}| \ge (b/a)^{3/4}} W(\vec{x}) dx \right)$$

for some constant $C > 0$. To bound the last term relative to the first one, we can use the lower bound of Lemma 3 to conclude that

$$\left\langle \Psi_{N,\omega,a}, H_{\omega,a}^N \Psi_{N,\omega,a} \right\rangle \le E_N^{\text{LLL}}(\omega, a) \left(1 + C \left[\frac{a}{b} + Nab^2 \right. \right.$$

$$\left. \left. + \frac{\kappa N^2 ab^3}{\min\{1, \kappa N^2\}} \left(1 + Nb^3 + \frac{b}{a} \int_{|\vec{x}| \ge (b/a)^{3/4}} W(\vec{x}) dx \right) \right] \right). \tag{28}$$

It remains to derive a lower bound on $\|\Psi_{N,\omega,a}\|$. Arguing as in [26] we can bound

$$\|\Psi_{N,\omega,a}\|^2 = \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} \prod_{1 \le i < j \le N} f_a(|\vec{x}_i - \vec{x}_j|)^2 |\Phi_{N,\omega,a}|^2$$

$$\geq 1 - \sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} (1 - f_a^2)(|\vec{x}_i - \vec{x}_j|)|\Phi_{N,\omega,a}|^2$$

$$\geq 1 - C N^2 \int_{\mathbb{R}^3} (1 - f_a^2)(|\vec{x}|) \frac{|\vec{x}|^4}{1 + |\vec{x}|^4} \, dx$$

$$- \left(\int_{\mathbb{R}^3} (1 - f_a^2) \right) \langle F_{N,\omega,a}, \mathcal{I}_N F_{N,\omega,a} \rangle_{\mathcal{B}_N}$$

where we have used again Lemma 2. For our choice of f_a in (27), it is easy to see [24, 25] that

$$\int_{\mathbb{R}^3} (1 - f_a^2) \leq 4\pi a b^2$$

and hence

$$\int_{\mathbb{R}^3} (1 - f_a^2)(|\vec{x}|) \frac{|\vec{x}|^4}{1 + |\vec{x}|^4} \, dx \leq 4\pi a b^6.$$

Finally, we can use Lemma 3 to bound $\langle F_{N,\omega,a}, \mathcal{I}_N F_{N,\omega,a} \rangle_{\mathcal{B}_N} \leq E_N^{\mathrm{LLL}}(\omega, a)/(4\pi a) \leq (2\pi)^{-3/2} N\kappa^{-1}$. This yields

$$\left\| \Psi_{N,\omega,a} \right\|^2 \geq 1 - CNab^2 \left(\frac{1}{\kappa} + Nb^4 \right) \tag{29}$$

for an appropriate constant $C > 0$.

Combining (28) and (29), the choice $b = 2\kappa^{-1/4} N^{-1/2}$ leads to the desired inequality (9).

Remark 2 If we had a lower bound

$$E_N^{\mathrm{LLL}}(\omega, a) \geq cNa\kappa^{-1},$$

as is expected for $\kappa \gtrsim N^{-1}$, the upper bound of Theorem 1 could be somewhat improved. In (28) $\min\{1, \kappa N^2\}$ could be replaced by $N \min\{1, \kappa N\}$ in the denominator, and the optimal choice of b would then be $b = (\kappa N)^{-1/3}$. The main error term would then be of the order $a\kappa^{1/3} N^{1/3}$ instead of $a\kappa^{1/4} N^{1/2}$.

Step 2: Lower Bound As a first step, we shall replace W_a by the finite range potential $W_{a,R_0} = W_a \chi(|\vec{x}| \leq R_0)$ for some R_0 to be chosen later. Since W_a is assumed to be non-negative, this is legitimate for a lower bound. We denote by $a(R_0)$ the scattering length of W_{a,R_0}. If R_0 is large enough compared to a, we will have $a(R_0) \simeq a$. Indeed, let us recall that [24]

$$4\pi a \geq 4\pi a(R_0) = \int_{\mathbb{R}^3} W_{a,R_0} f_{a,R_0} \geq \int_{\mathbb{R}^3} W_{a,R_0} f_a \geq 4\pi a - \int_{|\vec{x}| \geq R_0} W_a \tag{30}$$

where $f_a \leq f_{a,R_0}$ are the solutions of the zero-scattering equations corresponding to W_a and W_{a,R_0}, respectively.

We continue with a lemma inspired by a method of Dyson [16]. The key idea is to replace the "hard" interaction potential W_a by a softer one using parts of the kinetic energy, with this softer potential being close to $4\pi a\delta$ when projected to the LLL. Note that this step is essential, it is not possible to project the original W_a to the LLL level. This would also lead

to a δ interaction for small a, but with the wrong coupling constant $\int W_a$ instead of $4\pi a$ (see Remark 1). Compared to previous studies [22, 23, 28, 38] where a similar strategy has been applied, the main new difficulty comes from the fact that the effective kinetic energy $h_0 = (|\vec{p} - \vec{e}_3 \times \vec{x}|^2 + |x^3|^2 - 3)/2$ is not positive *locally*, i.e., on a domain with Neumann boundary conditions, but is positive only on the whole of $\mathbb{R}^3$. To circumvent this problem, we shall rewrite $\langle \Psi, h_0 \Psi \rangle$ for $\Psi \in L^2(\mathbb{R}^3)$ as

$$\langle \Psi, h_0 \Psi \rangle = \frac{1}{2} \int_{\mathbb{R}^3} e^{-|\vec{x}|^2} \left(|\partial_{x^3} \psi(\vec{x})|^2 + |\partial_{\bar{z}} \psi(\vec{x})|^2 \right) dx \tag{31}$$

with $\psi(\vec{x}) = e^{|\vec{x}|^2/2} \Psi(\vec{x})$ and $\partial_{\bar{z}} = \partial_{x^1} + i \partial_{x^2}$. The integrand on the right side is now positive but contains no derivatives with respect to z and is hence weaker than $|\vec{\nabla} \psi|^2$. Nevertheless we shall show in the next lemma that it is still strong enough to accomplish the goal of replacing W_a by a softer potential for a lower bound. The resulting "potential" turns out not be a potential in the usual sense of a multiplication operator, but rather is a non-local operator which has the property that its projection to the LLL is proportional to a δ-function, however.

Lemma 4 (Dyson-Type Inequality) *Let* $\vec{y} = (s, y^3) \in \mathbb{R}^3$. *For* $R > R_0$, *we have for all* ψ

$$\int_{|\vec{x}-\vec{y}| \leq R} e^{-|\vec{x}|^2} \left(|\partial_{x^3} \psi(\vec{x})|^2 + |\partial_{\bar{z}} \psi(\vec{x})|^2 + W_a(\vec{x} - \vec{y})|\psi(\vec{x})|^2 \right) dx$$

$$\geq 4\pi a(R_0) e^{-(|y^3|+R)^2 + |s|^2} \left| \frac{1}{4\pi R^2} \int_{|\vec{x}-\vec{y}|=R} e^{-\bar{s}z} \psi(\vec{x}) \, dx \right|^2. \tag{32}$$

Note that if $\psi \in \mathcal{B}_1$, one has

$$\frac{1}{4\pi R^2} \int_{|\vec{x}-\vec{y}|=R} e^{-\bar{s}z} \psi(\vec{x}) \, dx = e^{-|s|^2} \psi(\vec{y}),$$

and the right side of (32) equals $4\pi a(R_0) e^{-(|y^3|+R)^2 - |s|^2} |\psi(\vec{y})|^2$ which is precisely $4\pi a(R_0) \langle \psi, \delta_y \psi \rangle_{\mathcal{B}_1}$ when $R = 0$.

Proof of Lemma 4 Let $g(\vec{x}) = e^{-\bar{s}z} \psi(\vec{x} + \vec{y})$. Using that $W_a \geq W_{a,R_0}$, we have to show that

$$\int_{|\vec{x}| \leq R} e^{-|z|^2 - (x^3 + y^3)^2} \left(|\partial_{x^3} g(\vec{x})|^2 + |\partial_{\bar{z}} g(\vec{x})|^2 + W_{a,R_0}(\vec{x})|g(\vec{x})|^2 \right) dx$$

$$\geq 4\pi a(R_0) e^{-(|y^3|+R)^2} \left| \frac{1}{4\pi R^2} \int_{|\vec{x}|=R} g(\vec{x}) \, dx \right|^2. \tag{33}$$

Since $|z|^2 + (x^3 + y^3)^2 \leq (|y^3| + R)^2$ in the integrand on the left, this will follow if we can show that

$$\int_{|\vec{x}| \leq R} \left(|\partial_{x^3} g(\vec{x})|^2 + |\partial_{\bar{z}} g(\vec{x})|^2 + W_{a,R_0}(\vec{x})|g(\vec{x})|^2 \right) dx$$

$$\geq 4\pi a(R_0) \left| \frac{1}{4\pi R^2} \int_{|\vec{x}|=R} g(\vec{x}) \, dx \right|^2. \tag{34}$$

Let now f_{a,R_0} be the solution of the zero-energy scattering equation $-\Delta f_{a,R_0} + W_{a,R_0} f_{a,R_0} = 0$, subject to the normalization $\lim_{|\vec{x}| \to \infty} f_{a,R_0}(\vec{x}) = 1$. Since f_{a,R_0} is real-valued, $|\partial_{\bar{z}} f_{a,R_0}|^2 = |\partial_{x^1} f_{a,R_0}|^2 + |\partial_{x^2} f_{a,R_0}|^2$, and hence

$$\int_{|\vec{x}| \leq R} \left(|\partial_{x^3} f_{a,R_0}(\vec{x})|^2 + |\partial_{\bar{z}} f_{a,R_0}(\vec{x})|^2 + W_{a,R_0}(\vec{x})|f_{a,R_0}(\vec{x})|^2 \right) dx \leq 4\pi a(R_0).$$

The Cauchy-Schwarz inequality implies that

$$\int_{|\vec{x}| \leq R} \left(|\partial_{x^3} g(\vec{x})|^2 + |\partial_{\bar{z}} g(\vec{x})|^2 + W_{a,R_0}(\vec{x})|g(\vec{x})|^2 \right) dx$$

$$\geq \frac{1}{4\pi a(R_0)} \left| \int_{|\vec{x}| \leq R} \left(\partial_{x^3} f_{a,R_0} \partial_{x^3} g + \partial_z f_{a,R_0} \partial_{\bar{z}} g + W_{a,R_0} f_{a,R_0} g \right) \right|^2. \tag{35}$$

Using partial integration, the zero-energy scattering equation as well as the fact that $|\nabla f_{a,R_0}(\vec{x})| = a/|\vec{x}|$ for $|\vec{x}| \geq R_0$ this yields (34). $\qquad\square$

As an immediate corollary, we see that for any non-negative function ρ supported on $[R_0, R]$ with $\int_{R_0}^R \rho \leq 1$,

$$\int_{|\vec{x} - \vec{y}| \leq R} e^{-|\vec{x}|^2} \left(|\partial_{x^3} \psi(\vec{x})|^2 + |\partial_{\bar{z}} \psi(\vec{x})|^2 + W_a(\vec{x} - \vec{y})|\psi(\vec{x})|^2 \right) dx$$

$$\geq 4\pi a(R_0) e^{-(|y^3|+R)^2+|s|^2} \int_{R_0}^R dr \rho(r) \left| \frac{1}{4\pi r^2} \int_{|\vec{x}-\vec{y}|=r} e^{-\bar{s}z} \psi \right|^2. \tag{36}$$

We shall apply this inequality to the Hamiltonian $H_{\omega,a}^N$, for each particle separately, considering the other $N-1$ particles as fixed. Consider first particle one, and assume that all particles $k \geq 2$ are located at a distance $\geq 2R$ from each other, i.e., that $|\vec{x}_k - \vec{x}_\ell| \geq 2R$ for all $k, \ell = 2, \ldots, N$. Then we get, for all functions $F(\vec{x}_1, \ldots, \vec{x}_N)$,

$$\int_{\mathbb{R}^3} dx_1 e^{-|\vec{x}_1|^2} \left(|\partial_{x_1^3} F(\vec{x}_1, \ldots, \vec{x}_N)|^2 + |\partial_{\bar{z}_1} F(\vec{x}_1, \ldots, \vec{x}_N)|^2 \right.$$

$$\left. + \sum_{j=2}^N W_a(\vec{x}_j - \vec{x}_1)|F(\vec{x}_1, \ldots, \vec{x}_N)|^2 \right)$$

$$\geq 4\pi a(R_0) \sum_{j=2}^N e^{-(|x_j^3|+R)^2+|z_j|^2} \int_{R_0}^R dr \rho(r) \left| \frac{1}{4\pi r^2} \int_{|\vec{x}_1-\vec{x}_j|=r} e^{-\bar{z}_j z_1} F(\vec{x}_1, \ldots, \vec{x}_N) dx_1 \right|^2.$$

In general, we can get the same bound if we only retain on the right side the $\vec{x}_j$'s for $j = 2, \ldots, N$ which are at a distance $\geq 2R$ from all the others. Using (26) and (31), we conclude that, for any $0 \leq \theta \leq 1$,

$$H_{\omega,a}^N \geq \sum_{j=1}^N \left(\theta \, (h_0)_j + \omega(\vec{e}_3 \cdot \vec{L})_j \right) + 4\pi a(R_0)(1-\theta) \sum_{1 \leq i < j \leq N} U_{ij}, \tag{37}$$

where the potential U_{ij} is defined as

$$
\langle \Psi, U_{12}\Psi \rangle
$$
$$
= \int dx_2 \cdots \int dx_N e^{-|\vec{x}_2|^2} e^{-(|x_2^3|+R)^2+|z_2|^2} e^{-\sum_{k=3}^N |\vec{x}_k|^2}
$$
$$
\times \prod_{k=3}^N \chi_{|\vec{x}_k - \vec{x}_2| \geq 2R} \int_{R_0}^R dr \rho(r) \left| \frac{1}{4\pi r^2} \int_{|\vec{x}_1 - \vec{x}_2|=r} e^{-\overline{z_2}z_1} F(\vec{x}_1, \ldots, \vec{x}_N) dx_1 \right|^2, \quad (38)
$$

with $F(\vec{x}_1, \ldots, \vec{x}_N) = \pi^{N/4} \Psi(\vec{x}_1, \ldots, \vec{x}_2) \prod_{j=1}^N e^{|\vec{x}_j|^2/2}$ and $\vec{x}_1 = (z_1, x_1^3)$, $\vec{x}_2 = (z_2, x_2^3)$.

The new potential $\sum_{i<j} U_{ij}$ is a complicated N-body term which has the advantage of being bounded, however. To be precise, the following bound holds.

Lemma 5 *We have*

$$
\|U_{12}\| \leq \sup_r \frac{\rho(r)}{4\pi r^2}. \quad (39)
$$

Proof Applying the Cauchy-Schwarz inequality to the $\vec{x}_1$ integration, we see that

$$
\left| \int_{|\vec{x}_1 - \vec{x}_2|=r} dx_1 e^{-\overline{z_2}z_1} \psi \right|^2 \leq \int_{|\vec{x}_1 - \vec{x}_2|=r} dx_1 e^{-|\vec{x}_1|^2} |\psi|^2 \int_{|\vec{x}_1 - \vec{x}_2|=r} dx_1 e^{|\vec{x}_1|^2 - 2\operatorname{Re}\overline{z_2}z_1}.
$$

It is easy to check that $-(|x_2^3|+R)^2 + |z_2|^2 + |\vec{x}_1|^2 - 2\operatorname{Re}\overline{z_2}z_1 \leq 0$ for $|\vec{x}_1 - \vec{x}_2| \leq R$. Hence

$$
\langle \Psi, U_{12}\Psi \rangle \leq \int \cdots \int |\Psi(\vec{x}_1, \ldots, \vec{x}_N)|^2 \frac{\rho(|\vec{x}_1 - \vec{x}_2|)}{4\pi |\vec{x}_1 - \vec{x}_2|^2} dx_1 \cdots dx_N.
$$

This proves the claim. $\qquad\square$

In order to minimize the right side of (39), we shall choose

$$
\rho(r) = \frac{3r^2}{R^3 - R_0^3} \quad \text{for } R_0 \leq r \leq R. \quad (40)
$$

Note that $\int_{R_0}^R \rho(r)dr = 1$.

We shall now apply a standard perturbation theory argument to our Hamiltonian. Let P denote the orthogonal projection onto $\mathfrak{H}_N$ (the LLL for all the N particles), and let $Q = 1 - P$. Let

$$
A = \sum_{j=1}^N \left(\theta\,(h_0)_j + \omega(\vec{e}_3 \cdot \vec{L})_j \right)
$$

and

$$
B = 4\pi a(R_0)(1-\theta) \sum_{i<j} U_{ij}
$$

so that the right side of (37) equals $A + B$. We have $A = PAP + QAQ$. Since B is positive we can use the Cauchy-Schwarz inequality to get the lower bound

$$
B \geq (1-\delta)PBP + \left(1 - \delta^{-1}\right) QBQ
$$

for any $0 < \delta < 1$. Using Lemma 5 with the choice (40) for ρ we see that

$$QBQ \le Q\frac{6\pi a N(N-1)}{R^3 - R_0^3}.$$

(Recall that $a(R_0) \le a$.) Moreover, for $\theta > \omega$,

$$QAQ \ge Q(\theta - \omega).$$

We therefore conclude that

$$A + B \ge (1 - \delta)P(A + B)P + Q\left(\theta - \omega - \frac{6\pi a N^2}{\delta(R^3 - R_0^3)}\right). \tag{41}$$

In particular,

$$E_N(\omega, a) \ge \min\left\{(1 - \delta)\inf\sigma(A + B)\restriction_{\mathfrak{H}_N}, \theta - \omega - \frac{6\pi a N^2}{\delta(R^3 - R_0^3)}\right\}$$

and it remains to study $A + B$ restricted to $\mathfrak{H}_N$.

For $\Psi(\vec{x}_1, \ldots, \vec{x}_N) = \pi^{-N/4}\prod_{j=1}^{N} e^{-|\vec{x}_j|^2/2}F(z_1, \ldots, z_N)$ a bosonic function in $\mathfrak{H}_N$, we have

$$\langle \Psi, A\Psi \rangle = \omega \langle F, \mathcal{L}_N F \rangle_{\mathcal{B}_N}$$

by (11), and

$$\langle \Psi, B\Psi \rangle = 2\pi a(R_0)N(N-1)(1-\theta)\int dx_2 e^{|\vec{x}_2|^2 - (|x_2^3| + R)^2}$$

$$\times \int \prod_{j \ge 3} \chi_{|\vec{x}_j - \vec{x}_2| \ge 2R}\, dx_j |\Psi(\vec{x}_2, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_N)|^2 \tag{42}$$

where we have used that F is analytic in z_1 and z_2. For a lower bound, we use

$$\prod_{j \ge 3}\chi_{|\vec{x}_j - \vec{x}_2| \ge 2R} \ge 1 - \sum_{j \ge 3}\chi_{|\vec{x}_j - \vec{x}_2| \le 2R}.$$

Letting

$$t_R = \frac{2}{\sqrt{\pi}}\int_0^\infty dt\, e^{-(t+R)^2} \ge 1 - \frac{2R}{\sqrt{\pi}},$$

we finally get

$$\langle \Psi, B\Psi \rangle \ge 4\pi a(R_0)(1-\theta)t_R\left\langle F, \left(\sum_{1 \le i < j \le N}\delta_{ij}\right)F\right\rangle_{\mathcal{B}_N}$$

$$- 2\pi a(R_0)N^2(N-1)(1-\theta)$$

$$\times \int dx_2 \cdots \int dx_N \chi_{|\vec{x}_3 - \vec{x}_2| \le 2R}|\Psi(\vec{x}_2, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_N)|^2.$$

By Carlen's inequality (18) with $p = 0$ and $\vec{x}_2, \vec{x}_4, \ldots, \vec{x}_N$ fixed, we have

$$|\Psi(\vec{x}_2, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_N)|^2 \leq C \int_{\mathbb{R}^3} |\Psi(\vec{x}_2, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_N)|^2 \, dx_3,$$

therefore

$$\int_{\mathbb{R}^3} dx_3 \chi_{|\vec{x}_3 - \vec{x}_2| \leq 2R} |\Psi(\vec{x}_2, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_N)|^2 \leq CR^3 \int_{\mathbb{R}^3} d\vec{x}_3 |\Psi(\vec{x}_2, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_N)|^2.$$

We conclude that

$$(A + B) \upharpoonright_{\mathfrak{H}_N} \geq \frac{(1 - \theta)\left(t_R - CR^3 N\right) a(R_0)}{a} E_N^{\text{LLL}}(\omega, a).$$

Our final inequality is therefore

$$E_N(\omega, a) \geq \min \left\{ \frac{(1 - \delta)(1 - \theta)(t_R - CR^3 N) a(R_0)}{a} E_N^{\text{LLL}}(\omega, a), \right.$$

$$\left. \theta - \omega - \frac{6\pi a N^2}{\delta(R^3 - R_0^3)} \right\}. \tag{43}$$

We now optimize constants. Recall from (24) that $E_N^{\text{LLL}}(\omega, a) \leq \kappa^{-1} a N$. We choose $R_0 = a^{1/9}$, $R^3 = 2R_0^3$, $\theta = \omega + 2a^{1/3}N$ and $\delta = 6\pi a^{1/3} N$ such that

$$\theta - \omega - \frac{6\pi a N^2}{\delta(R^3 - R_0^3)} = N a^{1/3}.$$

Assuming that $\kappa \geq a^{2/3}$ this expression is greater than $E_N^{\text{LLL}}(\omega, a)$. Recalling (30) the final result is then

$$E_N(\omega, a) \geq E_N^{\text{LLL}}(\omega, a) \left(1 - \omega - \frac{1}{4\pi} \int_{|\vec{x}| \geq a^{-8/9}} W(\vec{x}) dx - C\left[a^{1/3} N + a^{1/9} \right] \right). \tag{44}$$

If $\kappa < a^{2/3}$, the choice $R_0 = \kappa^{1/6}$, $R^3 = 2R_0^3$, $\theta = \omega + 2N a \kappa^{-1}$ and $\delta = 6\pi N \kappa^{1/2}$ yields the desired bound. This completes the proof of Theorem 1. $\qquad \square$

3.3 Proof of Theorem 2

The proof is a simple consequence of the bounds in the previous subsection, together with the right choice of parameters. If we choose R_0, R and δ as above, but θ to be bigger than the previous choice by an amount $\theta' \geq 0$, we conclude that

$$H_{\omega,a}^N \geq \left(1 - \theta' - C\left[\frac{N a^{1/3}}{r} + a^{1/9} r \right] - \frac{1}{4\pi} \int_{|\vec{x}| \geq r^{1/6} a^{-8/9}} W \right) \Pi^* \tilde{H}_{\omega,a}^N \Pi$$

$$+ \left(\theta' + E_N^{\text{LLL}}(\omega, a) \right) Q \tag{45}$$

where $\Pi = \pi^{N/4} e^{\sum_{j=1}^N |\vec{x}_j|^2/2} P$ denotes the projection from $L^2(\mathbb{R}^{3N})$ onto $\mathcal{B}_N$, and $r = \min\{1, \kappa a^{-2/3}\}$ as in the statement of Theorem 1. Let $\Pi_1 := \pi^{N/4} e^{\sum_{j=1}^N |\vec{x}_j|^2/2} P_N(\kappa)$ denote the projection from $L^2(\mathbb{R}^{3N})$ onto the ground eigenspace of the operator $\mathcal{L}_N/N^2 +$

$4\pi\kappa\mathcal{I}_N/N$, and $\gamma_N(\kappa) = (\lambda_2 - \lambda_1)/\lambda_1 > 0$ where λ_1 and λ_2 are the first and second eigenvalues of this operator, respectively. Introducing $\Pi_2 = \Pi - \Pi_1$, we have

$$\Pi^* \tilde{H}_{\omega,a}^N \Pi = \Pi_1^* \tilde{H}_{\omega,a}^N \Pi_1 + \Pi_2^* \tilde{H}_{\omega,a}^N \Pi_2 \geq E_N^{\mathrm{LLL}}(\omega,a) P_N(\kappa) + \Pi_2^* \tilde{H}_{\omega,a}^N \Pi_2.$$

Note that

$$\Pi_2^* \tilde{H}_{\omega,a}^N \Pi_2 = N^2 \omega \Pi_2^* \left(\frac{\mathcal{L}_N}{N^2} + 4\pi \frac{a}{N\omega} \frac{\mathcal{I}_N}{N} \right) \Pi_2$$

$$\geq N^2 \omega \left(1 - \left| \frac{a}{N\omega\kappa} - 1 \right| \right) \Pi_2^* \left(\frac{\mathcal{L}_N}{N^2} + 4\pi\kappa \frac{\mathcal{I}_N}{N} \right) \Pi_2$$

$$\geq E_N^{\mathrm{LLL}}(\omega,a)(1 + \gamma_N(\kappa)) \left(1 - \left| \frac{a}{N\omega\kappa} - 1 \right| \right) \left(1 - \left| \frac{N\omega\kappa}{a} - 1 \right| \right) \Pi_2^* \Pi_2.$$

For a, ω, and $\kappa - a/(N\omega)$ small enough, we obtain

$$H_{\omega,a}^N \geq E_N^{\mathrm{LLL}}(\omega,a) \left(1 - \theta' - C \left[\frac{Na^{1/3}}{r} + a^{1/9}r \right] - \frac{1}{4\pi} \int_{|\vec{x}| \geq r^{1/6} a^{-8/9}} W \right)$$

$$+ E_N^{\mathrm{LLL}}(\omega,a) \frac{\gamma_N(\kappa)}{2} \Pi_2^* \Pi_2 + \theta' Q.$$

Combined with the upper bound (9), this clearly yields $\|\Pi_2 \Psi_{\omega,a}^N\| \to 0$ and $\|Q\Psi_{\omega,a}^N\| \to 0$, as was claimed. □

References

1. Aftalion, A., Blanc, X.: Vortex lattices in rotating Bose–Einstein condensates. SIAM J. Math. Anal. **38**, 874–893 (2006)
2. Aftalion, A., Blanc, X.: Reduced energy functionals for a three-dimensional fast rotating Bose-Einstein condensates. Ann. H. Poincaré **25**, 339–355 (2008)
3. Aftalion, A., Blanc, X., Dalibard, J.: Vortex patterns in a fast rotating Bose-Einstein condensate. Phys. Rev. A **71**, 023611 (2005)
4. Aftalion, A., Blanc, X., Lewin, M.: Unpublished
5. Aftalion, A., Blanc, X., Nier, F.: Lowest Landau level functional and Bargmann spaces for Bose-Einstein condensates. J. Funct. Anal. **241**, 661–702 (2006)
6. Baksmaty, L.O., Yannouleas, C., Landman, U.: Rapidly rotating boson molecules with long- or short-range repulsion: an exact diagonalization study. Phys. Rev. A **75**, 023620 (2007)
7. Bargmann, V.: On the representations of the rotation group. Rev. Mod. Phys. **34**, 829 (1962)
8. Bertsch, G., Papenbrock, T.: Yrast line for weakly interacting trapped bosons. Phys. Rev. Lett. **83**, 5412–5414 (1999)
9. Butts, D.A., Rokhsar, D.S.: Predicted signatures of rotating Bose-Einstein condensates. Nature **397**, 327–329 (1999)
10. Carlen, E.: Some integral identities and inequalities for entire functions and their application to the coherent state transform. J. Funct. Anal. **97**, 231–249 (1991)
11. Cooper, N.R.: Rapidly rotating atomic gases. Adv. Phys. **57**, 539–616 (2008)
12. Cooper, N.R., Komineas, S., Read, N.: Vortex lattices in the lowest Landau level for confined Bose-Einstein condensates. Phys. Rev. A **70**, 033604 (2004)
13. Cooper, N.R., Wilkin, N.K.: Composite fermion description of rotating Bose-Einstein condensates. Phys. Rev. B **60**, R16279–R16282 (1999)

14. Cooper, N.R., Wilkin, N.K., Gunn, J.M.F.: Quantum phases of vortices in rotating Bose-Einstein condensates. Phys. Rev. Lett. **87**, 120405 (2001)
15. Dalfovo, F., Giorgini, S., Pitaevskii, L.P., Stringari, S.: Theory of Bose-Einstein condensation in trapped gases. Rev. Mod. Phys. **71**, 463–512 (1999)
16. Dyson, F.J.: Ground-state energy of a hard-sphere gas. Phys. Rev. **106**, 20–26 (1957)
17. Girvin, S., Jach, T.: Formalism for the quantum Hall effect: Hilbert space of analytic functions. Phys. Rev. B **29**, 5617–5625 (1984)
18. Gross, E.: Structure of a quantized vortex in boson systems. Nuovo Cimento **20**, 454–477 (1961)
19. Hussein, M., Vorov, O.: Generalized yrast states of a Bose-Einstein condensate in a harmonic trap for a universality class of interactions. Phys. Rev. A **65**, 035603 (2002)
20. Laughlin, R.B.: Anomalous quantum hall effect: An incompressible quantum fluid with fractionally charged excitations. Phys. Rev. Lett. **50**, 1395–1398 (1983)
21. Lieb, E.H., Seiringer, R.: Proof of Bose-Einstein condensation for dilute trapped gases. Phys. Rev. Lett. **88**, 170409 (2002)
22. Lieb, E.H., Seiringer, R.: Derivation of the Gross-Pitaevskii equation for rotating Bose gases. Commun. Math. Phys. **264**, 505–537 (2006)
23. Lieb, E.H., Seiringer, R., Solovej, J.P.: Ground-state energy of the low-density Fermi gas. Phys. Rev. A **71**, 053605 (2005)
24. Lieb, E.H., Seiringer, R., Solovej, J.P., Yngvason, J.: The Mathematics of the Bose Gas and Its Condensation, Oberwolfach Seminars. Birkhäuser, Basel (2005)
25. Lieb, E.H., Seiringer, R., Yngvason, J.: Bosons in a trap: a rigorous derivation of the Gross-Pitaevskii energy functional. Phys. Rev. A **61**, 043602 (2000)
26. Lieb, E.H., Seiringer, R., Yngvason, J.: One-dimensional behavior of dilute, trapped Bose gases. Commun. Math. Phys. **244**, 347–393 (2004)
27. Lieb, E.H., Seiringer, R., Yngvason, J.: Yrast line of a rapidly rotating Bose gas: Gross-Pitaevskii regime. Phys. Rev. A **79**, 063626 (2009)
28. Lieb, E.H., Yngvason, J.: Ground state energy of the low density Bose gas. Phys. Rev. Lett. **80**, 2504–2507 (1998)
29. Mashkevich, S., Matveenko, S., Ouvry, S.: Exact results for the spectra of bosons and fermions with contact interaction. Nucl. Phys. B **763**, 431–444 (2007)
30. Morris, A.G., Feder, D.L.: Validity of the Lowest Landau Level approximation for rotating Bose gases. Phys. Rev. A **74**, 033605 (2006)
31. Mottelson, B.: Yrast spectra of weakly interacting Bose-Einstein condensates. Phys. Rev. Lett. **83**, 2695–2698 (1999)
32. Papenbrock, T., Bertsch, G.F.: Rotational spectra of weakly interacting Bose-Einstein condensates. Phys. Rev. A **63**, 023616 (2001)
33. Pitaevskii, L.P.: Vortex lines in an imperfect Bose gas. Zh. Eksp. Teor. Fiz. **40**, 646–651 (1961)
34. Regnault, N., Chang, C.C., Jolicoeur, T., Jain, J.K.: Composite fermion theory of rapidly rotating two-dimensional bosons. J. Phys. B **39**, S89–S99 (2006)
35. Regnault, N., Jolicoeur, T.: Quantum hall fractions for spinless bosons. Phys. Rev. B **69**, 235309 (2004)
36. Regnault, N., Jolicoeur, T.: Parafermionic states in rotating Bose-Einstein condensates. Phys. Rev. B **76**, 235324 (2007)
37. Seiringer, R.: Ground state asymptotics of a dilute, rotating gas. J. Phys. A **36**, 9755–9778 (2003)
38. Seiringer, R., Yin, J.: Ground state energy of the low density Hubbard model. J. Stat. Phys. **131**, 1139–1154 (2008)
39. Seiringer, R., Yin, J.: The Lieb-Liniger model as a limit of dilute bosons in three dimensions. Commun. Math. Phys. **284**, 459–479 (2008)
40. Viefers, S., Hansson, T.H., Reimann, S.M.: Bose condensates at high angular momenta. Phys. Rev. A **62**, 053604 (2000)